Notes on proarrow equipments

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Abstract

Proarrow equipments are one of the frameworks for formal category theory. The purpose of formal category theory is to abstract and synthesize the theory of ordinary categories. With the structure of a proarrow equipment, we can interpret many concepts in category theory — such as (co)limits, Cauchy completeness, and pointwise Kan extensions — in a 2-categorical way. In these notes, we review the theory of proarrow equipments as presented in [Woo82] and others.

Contents

1 Basics of proarrow equipments

1.1 Basic bicategory theory

In this subsection, we collect some basic facts and fix some notation on bicategories.

A *bicategory* is a notion of weak 2-category. Namely, a bicategory is like a 2-category such that the associativity and unity axioms hold *only* up to coherent isomorphism. Basic references of 2-categories and bicategories are [Bén67] and [JY21]. In the following, we will omit the coherent isomorphisms for simplicity.

Let $\mathcal{K}$ be a bicategory. Objects of Hom categories $\mathcal{K}(A, B)$ are called 1-*morphisms* or simply *morphisms* of $\mathcal{K}$, and morphisms of Hom categories are called 2-*morphisms* or 2-*cells*. We let $\mathcal{K}^{\mathrm{op}}$ denote the bicategory obtained by reversing morphisms of $\mathcal{K}$ and $\mathcal{K}^{\mathrm{co}}$ denote the bicategory obtained by reversing 2-cells of $\mathcal{K}$. Put $\mathcal{K}^{\mathrm{coop}} = (\mathcal{K}^{\mathrm{op}})^{\mathrm{co}} = (\mathcal{K}^{\mathrm{co}})^{\mathrm{op}}$, which is obtained by reversing both 1-morphisms and 2-morphisms of $\mathcal{K}$.

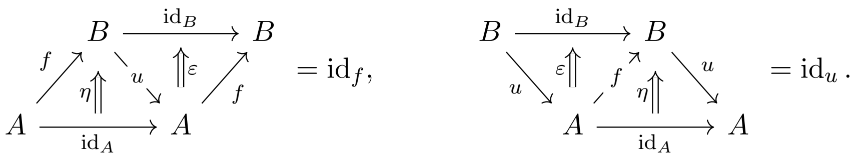
As in a 2-category, we can define equivalences and adjunctions in a bicategory as well.

*https://yuki-imamura.gitlab.io/notes.html.

Definition 1.1. Let $\mathcal{K}$ be a bicategory.

(1) An *equivalence* in $\mathcal{K}$ is a pair of morphisms $f\colon A \to B$ and $u\colon B \to A$ such that $u \circ f \cong \mathrm{id}_A$ and $u \circ f \cong \mathrm{id}_B$.

(2) An *adjunction* in $\mathcal{K}$ consists of a pair of morphisms $f\colon A \to B$ and $u\colon B \to A$ together with 2-cells $\eta\colon \mathrm{id}_A \Rightarrow u \circ f$ and $\varepsilon\colon f \circ u \Rightarrow \mathrm{id}_B$ that satisfy the following triangle identities:



In this case, we call f the *left adjoint*, u the *right adjoint*, η the *unit*, and ε the *counit*.

Example 1.2. (1) In the 2-category Cat of categories, an equivalence is just an equivalence of categories, and an adjunction is just a pair of adjoint functors.

(2) Consider the 2-category $\mathcal{V}$-Cat of enriched categories over a cosmos $\mathcal{V}$[1]. An adjunction in $\mathcal{V}$-Cat is equivalently a pair of $\mathcal{V}$-functors $F\colon \mathcal{A} \to \mathcal{B}$ and $G\colon \mathcal{B} \to \mathcal{A}$ together with a natural isomorphism $\mathcal{B}(FA, B) \cong \mathcal{A}(A, GB)$.

Now we introduce the notion of Kan extensions and Kan liftings.

Definition 1.3. Let $\mathcal{K}$ be a bicategory.

(1) Let $f\colon A \to B$ and $k\colon A \to D$ be morphisms of $\mathcal{K}$. A *left Kan extension* of f along k is a morphism $\mathrm{Lan}_k f\colon D \to B$ together with a 2-cell $\theta\colon f \Rightarrow \mathrm{Lan}_k f \circ k$ such that the map

$$\mathcal{K}(D,B)(\mathrm{Lan}_k f, h) \to \mathcal{K}(A,B)(f, h \circ k), \quad \chi \mapsto (\chi k) \circ \theta$$

is bijective for all morphisms $h\colon D \to B$. A left Kan extension in $\mathcal{K}^{\mathrm{co}}$, $\mathcal{K}^{\mathrm{op}}$, and $\mathcal{K}^{\mathrm{coop}}$ are called a *right Kan extension*, a *left Kan lifting*, and a *right Kan lifting*, and denoted by Ran, Lift, and Rift, respectively.

(2) Let $g\colon B \to C$ be another morphism of $\mathcal{K}$. Then we say that g *commutes with* the left Kan extension $\mathrm{Lan}_k f$ (resp. the right Kan extension $\mathrm{Ran}_k f$) if $g \circ \mathrm{Lan}_k f \cong \mathrm{Lan}_k(g \circ f)$ (resp. $g \circ \mathrm{Ran}_k f \cong \mathrm{Ran}_k(g \circ f)$).

(3) Also we say that g *commutes with* the left Kan liftings $\mathrm{Lift}_k f$ (resp. the right Kan liftings $\mathrm{Rift}_k f$) if $(\mathrm{Lift}_k f) \circ g \cong \mathrm{Lift}_k(f \circ g)$ (resp. $(\mathrm{Rift}_k f) \circ g \cong \mathrm{Rift}_k(f \circ g)$).

(4) A left Kan extension is called *absolute* if it is commuted with by all morphisms. We define absolute right Kan extensions and absolute Kan liftings in a similar way.

[1] A *(Bénabou) cosmos* means a complete and cocomplete symmetric monoidal closed category.

Proposition 1.4. Consider morphisms $f\colon A \to B$, $k\colon A \to D$, and $l\colon D \to E$ in a bicategory $\mathcal{K}$. If the left Kan extension $\operatorname{Lan}_k f$ exists, then we have an isomorphism

$$\operatorname{Lan}_l \operatorname{Lan}_k f \cong \operatorname{Lan}_{lk} f,$$

either side existing if the other does.

Proposition 1.5. Consider morphisms $f\colon A \to B$, $g\colon B \to A$ in a bicategory $\mathcal{K}$.

(1) For a 2-cell $\eta\colon \mathrm{id}_A \Rightarrow g \circ f$, the following are equivalent:

   (i) η is the unit of the adjunction $f \dashv g$.

   (ii) The pair (f, η) is the absolute left Kan lifting $\operatorname{Lift}_g \mathrm{id}_A$.

   (iii) The pair (f, η) is the left Kan lifting $\operatorname{Lift}_g \mathrm{id}_A$, and g commutes with it.

   (iv) The pair (g, η) is the absolute left Kan extension $\operatorname{Lan}_f \mathrm{id}_A$.

   (v) The pair (g, η) is the left Kan extension $\operatorname{Lan}_f \mathrm{id}_A$, and f commutes with it.

(2) For a 2-cell $\varepsilon\colon f \circ g \Rightarrow \mathrm{id}_B$, the following are equivalent:

   (i) ε is the counit of the adjunction $f \dashv g$.

   (ii) The pair (g, ε) is the absolute right Kan lifting $\operatorname{Rift}_f \mathrm{id}_B$.

   (iii) The pair (g, ε) is the right Kan lifting $\operatorname{Rift}_f \mathrm{id}_B$, and f commutes with it.

   (iv) The pair (f, ε) is the absolute right Kan extension $\operatorname{Ran}_g \mathrm{id}_B$.

   (v) The pair (f, ε) is the right Kan extension $\operatorname{Ran}_g \mathrm{id}_B$, and g commutes with it.

Proposition 1.6. Let $\mathcal{K}$ be a bicategory.

(1) A right Kan lifting along a left adjoint is a post-composite with the right adjoint: that is, for morphisms $h\colon A \to C$ and $f\colon B \to C$ where f has a right adjoint g, we have

$$\operatorname{Rift}_f h \cong g \circ h.$$

In particular, such a right Kan lifting is absolute.

(2) A right Kan extension along a right adjoint is a pre-composite with the left adjoint: that is, for morphisms $f\colon A \to B$ and $h\colon A \to C$ where f has a left adjoint g, we have

$$\operatorname{Ran}_f h \cong h \circ g.$$

In particular, such a right Kan extension is absolute.

Proof. This follows from Proposition 1.5. □

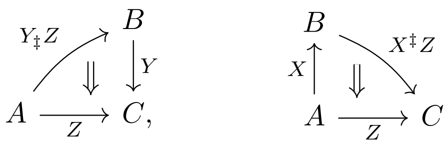
Proposition 1.7. Left adjoints commute with all right Kan liftings. Dually, right adjoints commute with all right Kan extensions.

Proof. Easy exercises on diagram chasing. □

Definition 1.8. A bicategory $\mathcal{M}$ is said to be *closed* if both the pre-composition functor and the post-composition functor have right adjoints. In other words, for morphisms $X\colon A \to B$, $Y\colon B \to C$, and $Z\colon A \to C$ in $\mathcal{M}$, there exist morphisms $Y_{\ddagger}Z$ and $X^{\ddagger}Z$ together with natural bijections

$$\mathcal{M}(A,C)(Y \circ X, Z) \cong \mathcal{M}(A,B)(X, Y_{\ddagger}Z),$$
$$\mathcal{M}(A,C)(Y \circ X, Z) \cong \mathcal{M}(B,C)(Y, X^{\ddagger}Z).$$

Proposition 1.9. Let $X\colon A \to B$, $Y\colon B \to C$, $Z\colon A \to C$ be morphisms in a bicategory $\mathcal{M}$ where $Y \circ -$ and $- \circ X$ have right adjoints $Y_{\ddagger}(-)$ and $X^{\ddagger}(-)$ respectively. Then the counits of these adjunctions



are respectively a right Kan lifting and a right Kan extension in $\mathcal{M}$. Furthermore, $\mathcal{M}$ being closed is equivalent to the existence of all right Kan liftings and right Kan extensions in $\mathcal{M}$.

Proof. This follows from the definitions. $\square$

Example 1.10 (The closed bicategory of profunctors)**.** For small categories $\mathcal{A}$ and $\mathcal{B}$, a *profunctor* $X\colon \mathcal{A} \nrightarrow \mathcal{B}$ is simply an ordinary functor $X\colon \mathcal{B}^{\mathrm{op}} \times \mathcal{A} \to \mathsf{Set}$. For profunctors $X\colon \mathcal{A} \nrightarrow \mathcal{B}$ and $Y\colon \mathcal{B} \nrightarrow \mathcal{C}$, their "composition" $Y \odot X$ is given by the coend

$$(Y \odot X)(c,a) = \int^{b \in \mathcal{B}} Y(c,b) \times X(b,a).$$

This composition makes profunctors into a bicategory Prof. The Hom set functor $\mathrm{Hom}_{\mathcal{A}}(-,-)\colon \mathcal{A}^{\mathrm{op}} \times \mathcal{A} \to \mathsf{Set}$ regarded as a profunctor $I_{\mathcal{A}}\colon \mathcal{A} \nrightarrow \mathcal{A}$ performs the identity morphism of $\mathcal{A}$ in Prof.

Furthermore, the bicategory Prof is closed. For profunctors $X\colon \mathcal{A} \nrightarrow \mathcal{B}$, $Y\colon \mathcal{B} \nrightarrow \mathcal{C}$, and $Z\colon \mathcal{A} \nrightarrow \mathcal{C}$, the profunctor $X^{\ddagger}Z\colon \mathcal{B} \nrightarrow \mathcal{C}$ is defined by

$$X^{\ddagger}Z(C,B) = \mathsf{Fun}(\mathcal{A},\mathsf{Set})(X(B,-),Z(C,-)) = \int_{A \in \mathcal{A}} \mathrm{Hom}\big(X(B,A), Z(C,A)\big),$$

and the profunctor $Y_{\ddagger}Z\colon \mathcal{A} \nrightarrow \mathcal{B}$ is defined by

$$Y_{\ddagger}Z(B,A) = \mathsf{Fun}(\mathcal{C}^{\mathrm{op}},\mathsf{Set})(Y(-,B),Z(-,A)) = \int_{C \in \mathcal{C}} \mathrm{Hom}\big(Y(C,B), Z(C,A)\big).$$

Then there are natural bijections

$$\mathsf{Prof}(\mathcal{A},\mathcal{C})(Y \odot X, Z) \cong \mathsf{Prof}(\mathcal{B},\mathcal{C})(Y, X^{\ddagger}Z),$$
$$\mathsf{Prof}(\mathcal{A},\mathcal{C})(Y \odot X, Z) \cong \mathsf{Prof}(\mathcal{A},\mathcal{B})(X, Y_{\ddagger}Z).$$

Proposition 1.11. Let $\mathcal{M}$ be a closed bicategory. Then for a morphism $X\colon A \to B$, we have

$$X \text{ is a left adjoint} \iff X \text{ commutes with all right Kan liftings,}$$
$$X \text{ is a right adjoint} \iff X \text{ commutes with all right Kan extensions.}$$

Proof. We show the first statement; the second is dual. The $(\Rightarrow)$ direction follows from Proposition 1.7. For $(\Leftarrow)$, since $\mathcal{M}$ is closed, the statement follows from Proposition 1.5. □

We also recall the notion of morphisms between bicategories.

Definition 1.12. Let $\mathcal{K}, \mathcal{L}$ be bicategories. A *lax functor* $\Phi\colon \mathcal{K} \to \mathcal{L}$ consists of

- a map $\Phi\colon \operatorname{ob}(\mathcal{K}) \to \operatorname{ob}(\mathcal{L})$,
- for each pair $A, B \in \mathcal{K}$ of objects, a functor $\Phi = \Phi_{AB}\colon \mathcal{K}(A,B) \to \mathcal{L}(\Phi(A), \Phi(B))$,
- for each object $A \in \mathcal{K}$, a 2-cell $\epsilon^A\colon \mathrm{id}_{\Phi(A)} \Rightarrow \Phi(\mathrm{id}_A)$, and
- for each triple $A, B, C \in \mathcal{K}$ of objects, a 2-cell $\mu^{A,B,C}_{g,f}\colon \Phi(g) \circ \Phi(f) \Rightarrow \Phi(g \circ f)$ natural in $f \in \mathcal{K}(A,B)$ and $g \in \mathcal{K}(B,C)$

such that these data satisfy the associativity and unity axioms (see [JY21, Definition 4.1.2] for the precise definition). We call ϵ^A the *lax unity constraint* and $\mu^{A,B,C}_{g,f}$ the *lax functoriality constraint.*

A lax functor $\Phi\colon \mathcal{K} \to \mathcal{L}$ is called *normal* (or *unitary*) if all ϵ^A are invertible, and a *pseudo-functor* if all ϵ^A and $\mu^{A,B,C}_{g,f}$ are invertible.

It is immediate to see that any pseudo-functor preserves equivalences and adjunctions, but this does not hold for a lax functor in general.

1.2 Proarrow equipments

Let $\mathcal{K}, \mathcal{M}$ be bicategories.

Definition 1.13 ([Woo82], [Woo85])**.** A pseudo-functor $(-)_*\colon \mathcal{K} \to \mathcal{M}$ is called a *proarrow equipment* if it satisfies:

(1) $(-)_*$ is bijective on objects,

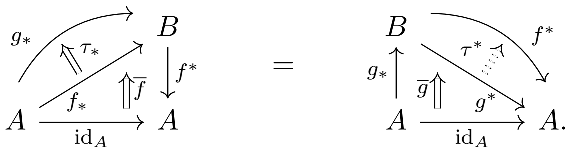
(2) $(-)_*$ is locally fully faithful, and

(3) for any 1-morphism f in $\mathcal{K}$, f_* has a right adjoint f^* in $\mathcal{M}$.

Note that in recent years, an equipment often refers to a special double category ([Shu08], [CS10]).

Since a proarrow equipment $(-)_*$ is bijective on objects, we henceforth identify the objects of $\mathcal{K}$ and $\mathcal{M}$, so $\operatorname{ob}(\mathcal{K}) = \operatorname{ob}(\mathcal{M})$.

For a proarrow equipment $(-)_* \colon \mathcal{K} \to \mathcal{M}$, given a morphism $f \colon A \to B$ in $\mathcal{K}$, there exists an adjunction $f_* \dashv f^*$ in $\mathcal{M}$. We write the unit of this adjunction by $\overline{f} \colon \mathrm{id}_A \Rightarrow f^* \circ f_*$. Since the unit is a left Kan extension, for any 2-morphism $\tau \colon f \Rightarrow g \colon A \to B$ in $\mathcal{K}$, one obtains a unique 2-morphism $\tau^* \colon g^* \Rightarrow f^* \colon B \to A$ in $\mathcal{M}$ such that



This correspondence defines a pseudo-functor

$$(-)^* \colon \mathcal{K}^{\mathrm{coop}} \to \mathcal{M},$$

which is locally fully faithful since $(-)_*$ is.

A morphism X in $\mathcal{M}$ is called *representable* if it can be written as $X \cong f_*$ for some morphism f in $\mathcal{K}$, and *corepresentable* if it can be written as $X \cong f^*$.

Example 1.14. The following are typical examples of proarrow equipments.

(1) Let Cat be the 2-category of small categories and Prof the bicategory of profunctors. For a functor $F \colon \mathcal{A} \to \mathcal{B}$, we define the profunctor $F_* \colon \mathcal{A} \nrightarrow \mathcal{B}$ by $F_* = \mathrm{Hom}_{\mathcal{B}}(-, F(-))$. It is known that F_* has a right adjoint $F^* = \mathrm{Hom}_{\mathcal{B}}(F(-), -) \colon \mathcal{B} \nrightarrow \mathcal{A}$ in Prof (see [Bor94, Proposition 7.9.1] or [Lor21, Remark 5.2.1] for example). Hence the mapping $F \mapsto F_*$ yields a proarrow equipment

$$(-)_* \colon \mathsf{Cat} \to \mathsf{Prof}.$$

(2) More generally, if $\mathcal{V}$ is a cosmos, then $\mathcal{V}$-enriched profunctors $X \colon \mathcal{A} \nrightarrow \mathcal{B}$ between $\mathcal{V}$-enriched small categories are defined similarly, and there exists a proarrow equipment of $\mathcal{V}$-enriched categories

$$(-)_* \colon \mathcal{V}\text{-}\mathsf{Cat} \to \mathcal{V}\text{-}\mathsf{Prof}, \quad F \mapsto \mathcal{B}(-, F(-)).$$

Example 1.15. (Other examples)

(1) Let $\mathcal{C}$ be a category with pullbacks, and let $\mathsf{Span}(\mathcal{C})$ denote the bicategory of spans in $\mathcal{C}$. Then the assignment $(f \colon c \to d) \mapsto (c \xleftarrow{\mathrm{id}_c} c \xrightarrow{f} d)$ defines a proarrow equipment

$$(-)_* \colon \mathcal{C} \to \mathsf{Span}(\mathcal{C}).$$

(2) If $\mathcal{S}$ is a finitely complete category, there exists a proarrow equipment

$$(-)_* \colon \mathsf{Cat}(\mathcal{S}) \to \mathsf{Prof}(\mathcal{S})$$

relating $\mathcal{S}$-internal categories and $\mathcal{S}$-internal profunctors.

(3) Let TopGeom denote the 2-category of elementary toposes and geometric morphisms (with morphisms reversed from the direction of left adjoints). Let TopLex denote the 2-category of elementary toposes and left exact functors. Then there exists a proarrow equipment

$$(-)_*\colon \mathsf{TopGeom}^{\mathrm{op}} \to \mathsf{TopLex}^{\mathrm{co}}$$

given by taking left adjoints of geometric morphisms.

(4) Similarly, let AbelGeom denote the 2-category of Abelian categories and geometric morphisms, and AbelLex the 2-category of Abelian categories and left exact functors. Then there exists a proarrow equipment

$$(-)_*\colon \mathsf{AbelGeom}^{\mathrm{op}} \to \mathsf{AbelLex}^{\mathrm{co}}$$

given by taking left adjoints of geometric morphisms.

Proposition 1.16 (Yoneda [Woo82, Proposition 3])**.** Let $(-)_*\colon \mathcal{K} \to \mathcal{M}$ be a proarrow equipment.

(1) For morphisms $f\colon B \to C$ in $\mathcal{K}$ and $Z\colon A \to C$ in $\mathcal{M}$, we have $\mathrm{Rift}_{f_*} Z \cong f^* \circ Z$.

(2) For morphisms $f\colon B \to A$ in $\mathcal{K}$ and $Z\colon A \to C$ in $\mathcal{M}$, we have $\mathrm{Ran}_{f^*} Z \cong Z \circ f_*$.

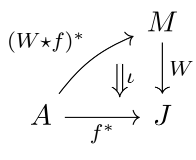
Proof. It is immediate from Proposition 1.6. □

Corollary 1.17. Let $(-)_*\colon \mathcal{K} \to \mathcal{M}$ be a proarrow equipment. For morphisms $f\colon B \to C, g\colon A \to C$ in $\mathcal{K}$, we have $\mathrm{Rift}_{f_*} g_* \cong f^* \circ g_* \cong \mathrm{Ran}_{g^*} f^*$.

1.3 The notion of (co)limits within a proarrow equipment

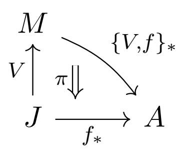
Let $(-)_*\colon \mathcal{K} \to \mathcal{M}$ be a proarrow equipment.

Definition 1.18 ([Woo82, §2])**.** For morphisms $f\colon J \to A$ in $\mathcal{K}$ and $W\colon M \to J$ in $\mathcal{M}$, the *W-weighted colimit* of f is a morphism $\mathrm{colim}^W f = W \star f\colon M \to A$ in $\mathcal{K}$ together with a right Kan lifting



in $\mathcal{M}$. If the W-weighted colimit of f exists, then $(W \star f)^* = \mathrm{Rift}_W f^*$. In other words, $W \star f$ exists if and only if the right Kan lift $\mathrm{Rift}_W f^*$ exists and is corepresentable.

Dually, for morphisms $f\colon J \to A$ in $\mathcal{K}$ and $V\colon J \to M$ in $\mathcal{M}$, the *V-weighted limit* of f is a morphism $\lim^V f = \{V, f\}\colon M \to A$ in $\mathcal{K}$ together with a right Kan extension



in $\mathcal{M}$. If the V-weighted limit of f exists, then $\{V, f\}_* = \mathrm{Ran}_V\, f_*$. In other words, $\{V, f\}$ exists if and only if the right Kan extension $\mathrm{Ran}_V\, f_*$ exists and is representable.

Example 1.19. Consider the proarrow equipment $\mathcal{V}\text{-}\mathsf{Cat} \to \mathcal{V}\text{-}\mathsf{Prof}$ of enriched categories, as in Example 1.14. Let $M = \mathcal{I}$ be the unit $\mathcal{V}$-category. For a $\mathcal{V}$-functor $F\colon \mathcal{J} \to \mathcal{A}$ and a $\mathcal{V}$-profunctor $W\colon \mathcal{I} \nrightarrow \mathcal{J}$, the right Kan lifting $\mathrm{Rift}_W\, F^*\colon \mathcal{A} \nrightarrow \mathcal{I}$ is given by

$$\mathrm{Rift}_W\, F^*(*, A) = \mathsf{Fun}(\mathcal{J}^{\mathrm{op}}, \mathcal{V})(W(-, *), \mathcal{A}(F-, A)).$$

Hence the W-weighted colimit of F in the sense of Definition 1.18 is precisely the enriched colimit of F weighted by the presheaf $W\colon \mathcal{J}^{\mathrm{op}} \cong \mathcal{J}^{\mathrm{op}} \otimes \mathcal{I} \to \mathcal{V}$ in the sense of [Kel82, §3.1].

Proposition 1.20 ([Woo82, Proposition 6])**.** For morphisms $f\colon J \to A$ in $\mathcal{K}$ and $V\colon N \to M$, $W\colon M \to A$ in $\mathcal{M}$, if the W-weighted colimit $W \star f$ exists, then we have an isomorphism

$$V \star (W \star f) \cong (W \circ V) \star f,$$

either side existing if the other does.

Dually, for suitable U, W, f, if the W-weighted limit $\{W, f\}$ exists, then we have an isomorphism

$$\{U, \{W, f\}\} \cong \{U \circ W, f\},$$

either side existing if the other does.

Proof. We prove the first half. In general, $\mathrm{Rift}_V(\mathrm{Rift}_W\, f^*) \cong \mathrm{Rift}_{W \circ V}\, f^*$ so we have

$$(V \star (W \star f))^* \cong \mathrm{Rift}_V(W \star f)^* \cong \mathrm{Rift}_V(\mathrm{Rift}_W\, f^*) \cong \mathrm{Rift}_{W \circ V}\, f^* \cong ((W \circ V) \star f)^*$$

and since $(-)^*$ is locally fully faithful, we get $V \star (W \star f) \cong (W \circ V) \star f$. □

Proposition 1.21 ([Woo82, Proposition 7])**.** For morphisms $f\colon J \to A$, $w\colon M \to J$ in $\mathcal{K}$, we have

$$w_* \star f \cong f \circ w \cong \{w^*, f\}.$$

Proof. By Proposition 1.16, we have

$$(f \circ w)^* \cong w^* \circ f^* \cong \mathrm{Rift}_{w_*}\, f^* = (w_* \star f)^*,$$

which implies $w_* \star f \cong f \circ w$. The other isomorphism is proved similarly. □

Definition 1.22. A morphism $g\colon A \to B$ in $\mathcal{K}$ is said to *preserve* a weighted colimit $W \star f$ if g^* commutes with the right Kan lifting $(W \star f)^* = \mathrm{Rift}_W\, f^*$.

Similarly, $g\colon A \to B$ is said to *preserve* a weighted limit $\{V, f\}$ if g_* commutes with the right Kan extension $\{V, f\}_* = \mathrm{Ran}_V\, f_*$.

Proposition 1.23 ([Woo82, Proposition 8])**.** Left adjoints preserve all weighted colimits. Dually, right adjoints preserve all weighted limits.

Proof. If a morphism f in $\mathcal{K}$ has a right adjoint u, then we get the adjunction $f^* \dashv u^*$ in $\mathcal{M}$, so by Proposition 1.7, f^* commutes with all right Kan liftings. Hence f preserves all weighted colimits in particular. □

Proposition 1.24 (Formal criterion for representability [Woo82, Proposition 9])**.** For morphisms $f\colon A \to B$ in $\mathcal{K}$ and $X\colon A \to B$ in $\mathcal{M}$, the following are equivalent:

(i) $X \cong f_*$.

(ii) The weighted colimit $X \star \mathrm{id}_B$ exists, $X \star \mathrm{id}_B \cong f$, and X commutes with all right Kan liftings.

(iii) The weighted colimit $X \star \mathrm{id}_B$ exists, $X \star \mathrm{id}_B \cong f$, and X commutes with the right Kan lifting $(X \star \mathrm{id}_B)^* = \mathrm{Rift}_X \mathrm{id}_B^*$.

Proof. (i) ⇒ (ii): By Proposition 1.21, we have

$$X \star \mathrm{id}_B \cong f_* \star \mathrm{id}_B \cong \mathrm{id}_B \circ f \cong f.$$

Also, since $X \cong f_*$ has a right adjoint, it commutes with all right Kan liftings.

(ii) ⇒ (iii): Clear.

(iii) ⇒ (i): From $X \star \mathrm{id}_B \cong f$, we have

$$f^* \cong (X \star \mathrm{id}_B)^* \cong \mathrm{Rift}_X \mathrm{id}_B^* \cong \mathrm{Rift}_X \mathrm{id}_B\,.$$

Since X commutes with this Kan lifting, Proposition 1.5 shows that there is an adjunction $X \dashv f^*$. As $f_* \dashv f^*$, we get $X \cong f_*$. □

We remark that for morphisms $f\colon A \to B$, $u\colon B \to A$ in $\mathcal{K}$, $f \dashv u$ holds in $\mathcal{K}$ if and only if $f^* \cong u_*$ holds in $\mathcal{M}$.

Corollary 1.25 (Formal adjoint arrow theorem [Woo82, Corollary 10])**.** For morphisms $f\colon A \to B$, $u\colon B \to A$ in $\mathcal{K}$, the following are equivalent:

(i) $f \dashv u$ holds in $\mathcal{K}$.

(ii) The weighted colimit $f^* \star \mathrm{id}_A$ exists, $f^* \star \mathrm{id}_A \cong u$, and f preserves all weighted colimits.

(iii) The weighted colimit $f^* \star \mathrm{id}_A$ exists, $f^* \star \mathrm{id}_A \cong u$, and f preserves the weighted colimit $f^* \star \mathrm{id}_A$.

Proof. This follows from Proposition 1.24 by taking $X = f^*\colon B \to A$. □

Definition 1.26. A morphism $f\colon A \to B$ in $\mathcal{K}$ is called *fully faithful* if the unit of the adjunction $f_* \dashv f^*$ is an isomorphism.

Example 1.27. In the proarrow equipment $\mathcal{V}\text{-}\mathsf{Cat} \to \mathcal{V}\text{-}\mathsf{Prof}$ of enriched categories over a cosmos $\mathcal{V}$, a $\mathcal{V}$-functor $F\colon \mathcal{A} \to \mathcal{B}$ is fully faithful in the above sense if and only if it is fully faithful in the enriched sense; that is, all $F_{AA'}\colon \mathcal{A}(A,A) \to \mathcal{B}(FA, FA')$ are isomorphisms for $A, A' \in \mathcal{A}$.

Proposition 1.28 (Part of [Woo82, Proposition 13])**.** For a left adjoint $f\colon A \to B$ in $\mathcal{K}$ with unit η, the following are equivalent.

(i) f is fully faithful.

(ii) η is an isomorphism.

Proof. If f has a right adjoint with unit η, then η_* is the unit of the adjunction $f_* \dashv f^*$. Thus the assertion follows, since $(-)_*$ is locally fully faithful. □

1.4 Absolute limits and Cauchy completeness

Most of the contents of this subsection are not included in [Woo82], but they are considered part of the folklore.

Let $(-)_*\colon \mathcal{K} \to \mathcal{M}$ be a proarrow equipment.

Definition 1.29. For morphisms $f\colon J \to A$ in $\mathcal{K}$ and $W\colon M \to J$ in $\mathcal{M}$, the W-weighted colimit $W \star f\colon M \to A$ of f is said to be *absolute* if it is preserved by every morphism $g\colon A \to B$ in $\mathcal{K}$. Similarly, the V-weighted limit $\{V, f\}\colon M \to A$ of f is said to be *absolute* if it is preserved by every morphism $g\colon A \to B$ in $\mathcal{K}$.

Proposition 1.30. Let $W\colon M \to J$ be a morphism in $\mathcal{M}$. If W has a right adjoint $V\colon J \to M$, then all W-weighted colimits are absolute. Similarly, if $W\colon J \to M$ has a left adjoint $U\colon M \to J$, then all W-weighted limits are absolute.

Proof. Suppose $f\colon J \to A$ is a morphism in $\mathcal{K}$ and the W-weighted colimit $W \star f\colon M \to A$ exists. Then $(W \star f)^* = \mathrm{Rift}_W f^*$. Since W is a left adjoint, Proposition 1.6 shows $\mathrm{Rift}_W f^* \cong V \circ f^*$. Hence for any $g\colon A \to B$ in $\mathcal{K}$, we have

$$\mathrm{Rift}_W f^* \circ g^* \cong V \circ f^* \circ g^* \cong V \circ (gf)^* \cong \mathrm{Rift}_W (gf)^*.$$

Thus g^* commutes with the right Kan lifting $\mathrm{Rift}_W f^*$, and g preserves the colimit $W \star f$. The case of weighted limits is similar. □

Proposition 1.31 (A generalization of [Gar14])**.** Let $W\colon M \to J$ be a morphism in $\mathcal{M}$ with a right adjoint $V\colon J \to M$. For morphisms $f\colon J \to A$ and $z\colon M \to A$ in $\mathcal{K}$, z is the (absolute) W-weighted colimit of f if and only if z is the (absolute) V-weighted limit of f.

Proof. Since $W \dashv V$, we see from Proposition 1.6

$$\mathrm{Rift}_W f^* \cong V \circ f^*, \qquad \mathrm{Ran}_V f_* \cong f_* \circ W.$$

In particular, we get an adjunction $\mathrm{Ran}_V f_* \dashv \mathrm{Rift}_W f^*$. Therefore, $\mathrm{Rift}_W f^*$ being corepresentable by z is equivalent to $\mathrm{Ran}_V f_*$ being representable by z. □

Definition 1.32. An object $A \in \mathcal{K}$ is said to be *Cauchy complete* if every left adjoint morphism $\Phi \colon D \to A$ in $\mathcal{M}$ is representable.

Proposition 1.33. For an object $A \in \mathcal{K}$, the following are equivalent:

(i) A is Cauchy complete.

(ii) A has all colimits weighted by left adjoints.

(iii) A has all limits weighted by right adjoints.

Proof. (i) ⇒ (ii): Let $W \colon D \to J$ be a left adjoint in $\mathcal{M}$ with right adjoint $V \colon J \to D$. For any morphism $f \colon J \to A$ in $\mathcal{K}$, by Proposition 1.31, the existence of $\mathrm{colim}^W f$ is equivalent to the existence of $\lim^V f$, which holds if and only if $\mathrm{Ran}_V f_* \cong f_* \circ W$ is representable. Since $f_* \circ W$ is a left adjoint in $\mathcal{M}$, it is representable by the Cauchy completeness of A.

(ii) ⇒ (i): Let $W \colon D \to A$ be a left adjoint in $\mathcal{M}$ with right adjoint $V \colon A \to D$. Since A has W-weighted colimits, in particular $g \coloneqq \mathrm{colim}^W \mathrm{id}_A$ exists. Then g^* is the right Kan lifting $\mathrm{Rift}_W \mathrm{id}_A^*$. Now $W \dashv V$ implies $V \cong g^*$, and hence $g_* \cong W$.

(ii) ⇔ (iii): This follows from Proposition 1.31. □

Therefore, by Proposition 1.30, an object is Cauchy complete if it has all absolute weighted colimits.

Remark 1.34. For the proarrow equipment $(-)_* \colon \mathcal{V}\text{-}\mathsf{Cat} \to \mathcal{V}\text{-}\mathsf{Prof}$ of enriched categories and enriched profunctors, the converse of Proposition 1.30 holds ([Str83]). That is, weights of absolute colimits have right adjoints; and hence an enriched category is Cauchy complete if and only if it has all absolute weighted colimits. It is not clear to the author whether this is the case for general proarrow equipments.

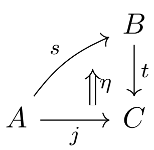
1.5 Relative adjunctions and pointwise Kan extensions

Let $(-)_* \colon \mathcal{K} \to \mathcal{M}$ be a proarrow equipment. Recall that the assignment that sends a morphism f in $\mathcal{K}$ to the right adjoint f^* of f_* induces a pseudo-functor

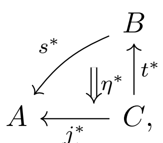
$$(-)^* \colon \mathcal{K}^{\mathrm{coop}} \to \mathcal{M}$$

that is bijective on objects and locally fully faithful.

Definition 1.35. A 2-cell



in $\mathcal{K}$ is said to be a *relative unit relative to* j if the 2-cell obtained by applying the pseudo-functor $(-)^*$ to it,



is a right Kan extension in $\mathcal{M}$. In this case, s is called a *relative left adjoint* of t.

Similarly, a 2-cell

$$\begin{array}{ccc} & \overset{s}{\nearrow} & B \\ & \Downarrow\varepsilon & \downarrow t \\ A & \xrightarrow[j]{} & C \end{array}$$

in $\mathcal{K}$ is said to be a *relative counit relative to j* if the 2-cell obtained by applying the pseudo-functor $(-)_*$ to it,

$$\begin{array}{ccc} & \overset{s_*}{\nearrow} & B \\ & \Downarrow\varepsilon_* & \downarrow t_* \\ A & \xrightarrow[j_*]{} & C, \end{array}$$

is a right Kan lifting in $\mathcal{M}$. In this case, s is called a *relative right adjoint* of t.

Proposition 1.36 ([Woo82, Proposition 11])**.** Relative adjunctions are absolute Kan liftings.

Proof. Consider a 2-cell in $\mathcal{K}$

$$\begin{array}{ccc} & \overset{s}{\nearrow} & B \\ & \Uparrow\eta & \downarrow t \\ A & \xrightarrow[j]{} & C \end{array}$$

that is a relative unit. We want to show that η is an absolute left Kan lifting in $\mathcal{K}$. Take an arbitrary 2-cell

$$\begin{array}{ccc} X & \xrightarrow{b} & B \\ a\downarrow & \Uparrow\chi & \downarrow t \\ A & \xrightarrow[j]{} & C \end{array}$$

in $\mathcal{K}$. Since the pseudo-functor $(-)^*\colon \mathcal{K}^{\mathrm{coop}} \to \mathcal{M}$ is locally fully faithful, this 2-cell corresponds bijectively to the 2-cell

$$\begin{array}{ccc} X & \xleftarrow{b^*} & B \\ a^*\uparrow & \Downarrow\chi^* & \uparrow t^* \\ A & \xleftarrow[j^*]{} & C \end{array}$$

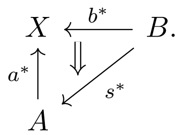
in $\mathcal{M}$. Since the counit $\underline{a}\colon a_* \circ a^* \Rightarrow \mathrm{id}_A$ of the adjunction $a_* \dashv a^*$ is an absolute right Kan lifting $\mathrm{Rift}_{a_*} \mathrm{id}_A$, this corresponds bijectively to the 2-cell

$$\begin{array}{ccc} X & \xleftarrow{b^*} & B \\ a_*\downarrow & \Downarrow & \uparrow t^* \\ A & \xleftarrow[j^*]{} & C. \end{array}$$

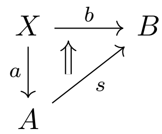
By assumption η^* is a right Kan extension, and so this corresponds bijectively to the 2-cell

$$\begin{array}{ccc} X & \xleftarrow{b^*} & B. \\ a_*\downarrow & \Downarrow & \swarrow s^* \\ A & & \end{array}$$

Once again, since the counit $\underline{a}\colon a_* \circ a^* \Rightarrow \mathrm{id}_A$ is an absolute right Kan lifting, this corresponds bijectively to the 2-cell



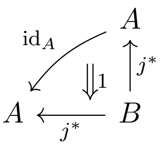
By the locally fully faithfulness of the pseudo-functor $(-)^*\colon \mathcal{K}^{\mathrm{coop}} \to \mathcal{M}$, this 2-cell in $\mathcal{M}$ corresponds bijectively to the 2-cell



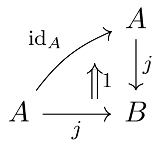
in $\mathcal{K}$. The inverse of these correspondences is obtained by pasting with η, and therefore η is an absolute left Kan lifting.

Similarly, we see that a relative counit is an absolute right Kan lifting. □

Example 1.37. Recall that a morphism $j\colon A \to B$ in $\mathcal{K}$ is said to be fully faithful if the unit $\overline{j}\colon \mathrm{id}_A \Rightarrow j^* \circ j_*$ of the adjunction in $\mathcal{M}$ is an isomorphism (see Definition 1.26). By Corollary 1.17, we have an isomorphism $j^* \circ j_* \cong \mathrm{Ran}_{j^*} j^*$. Therefore, j is fully faithful if and only if the 2-cell



is a right Kan extension in $\mathcal{M}$. In other words, this is equivalent to saying that the following 2-cell in $\mathcal{K}$

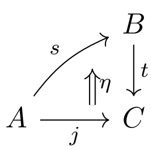


is a relative unit.

Remark 1.38. Consider the proarrow equipment $(-)_*\colon \mathcal{V}\text{-}\mathsf{Cat} \to \mathcal{V}\text{-}\mathsf{Prof}$ of enriched categories. Then the identity 2-cell $1\colon j \to j \circ \mathrm{id}$ becomes a relative unit if and only if j is fully faithful as an enriched functor. On the other hand, the fact that $1\colon j \to j \circ \mathrm{id}$ is an absolute left Kan lifting is equivalent to the underlying functor j_0 being fully faithful as an ordinary functor. This shows that the converse of Proposition 1.36 does not hold in general.

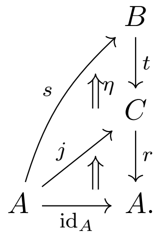
However, the following holds.

Proposition 1.39 ([Woo82, Proposition 12])**.** Consider a 2-cell in $\mathcal{K}$

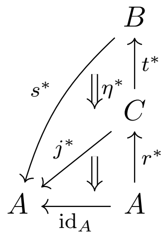


that is an absolute left Kan lifting. If either (1) j is a left adjoint or (2) t is a right adjoint, then η is a relative unit.

Proof. (1) Suppose that j is a left with right adjoint r. Then the unit $\mathrm{id}_A \Rightarrow r \circ j$ is an absolute left Kan lifting, and hence so is the composite 2-cell

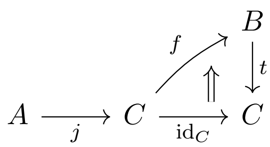


In particular, this 2-cell serves as the unit for an adjunction $s \dashv rt$. Since this implies $s^* \dashv (rt)^*$, the corresponding 2-cell in $\mathcal{M}$

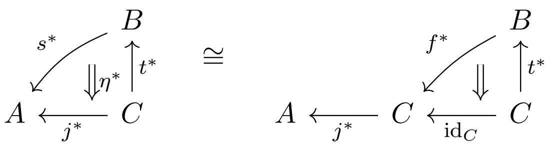


is an absolute right Kan extension. The lower triangle is the counit of the adjunction $j^* \dashv r^*$ and thus is also an absolute right Kan extension. It follows that η^* is a right Kan extension in $\mathcal{M}$, and therefore η is a relative unit.

(2) Suppose that t is a right adjoint with left adjoint f. Then the unit $\mathrm{id}_C \Rightarrow t \circ f$ is an absolute left Kan lifting, so the 2-cell



is also a left Kan lifting. Hence this 2-cell is isomorphic to η. It follows that we have the isomorphism



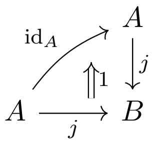
The right triangle on the right-hand side is the counit of the adjunction $f^* \dashv t^*$, so it is an absolute right Kan extension. Therefore, η^* is a right Kan extension in $\mathcal{M}$, and therefore η is a relative unit. □

Proposition 1.40 (Part of [Woo82, Proposition 13])**.** Let $j\colon A \to B$ be a morphism in $\mathcal{K}$ which has a left or right adjoint. Then the following are equivalent:

(i) j is fully faithful, in the sense that the unit $\bar{j}\colon \mathrm{id}_A \Rightarrow j^* \circ j_*$ of the adjunction $j_* \dashv j^*$ is an isomorphism.

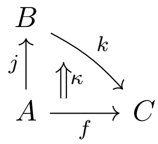
(ii) j is *representably full faithful*; that is, for all $X \in \mathcal{K}$, the post-composition functor $j \circ -: \mathcal{K}(X, A) \to \mathcal{K}(X, B)$ is fully faithful.

Proof. As seen in Example 1.37, the condition (i) is equivalent to saying that the identity 2-cell

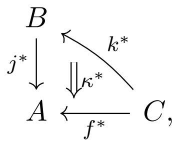


is a relative unit. On the other hand, the condition (ii) states that this 2-cell is an absolute left Kan lifting. Therefore, the assertion follows from Proposition 1.36 and Proposition 1.39. □

Definition 1.41. A 2-cell in $\mathcal{K}$

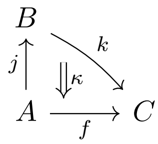


is said to be a *pointwise left Kan extension* of f along j if the 2-cell obtained by applying the pseudo-functor $(-)^*$ to it,

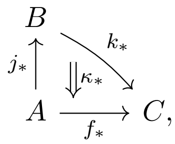


is a right Kan lifting in $\mathcal{M}$.

Similarly, a 2-cell in $\mathcal{K}$



is said to be a *pointwise right Kan extension* of f along j if 2-cell obtained by applying the pseudo-functor $(-)_*$ to it,



is a right Kan extension in $\mathcal{M}$.

Remark 1.42. In other words, a 2-cell $\kappa : f \Rightarrow k \circ j$ is a pointwise left Kan extension if and only if k is the j^*-weighted colimit of f, that is, $k \cong j^* \star f$.

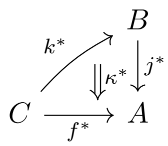
Similarly, 2-cell $\kappa : k \circ j \Rightarrow f$ is a pointwise right Kan extension if and only if k is j_*-weighted limit of f, that is, $k \cong \{j_*, f\}$.

Proposition 1.43. Pointwise Kan extensions are Kan extensions.

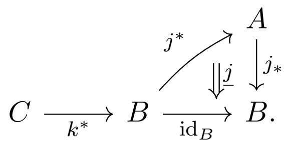
Proof. It follows immediately from the fact that the pseudo-functors $(-)^*, (-)_*$ are locally fully faithful. □

Proposition 1.44 ([Woo82, Proposition 14])**.** Let $\kappa \colon f \Rightarrow k \circ j$ be a pointwise Kan extension in $\mathcal{K}$. If j is fully faithful, then κ is an isomorphism.

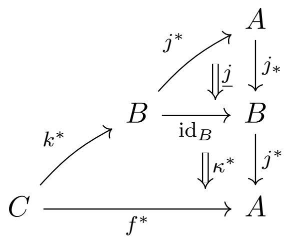
Proof. Suppose that κ is a pointwise left Kan extension so that



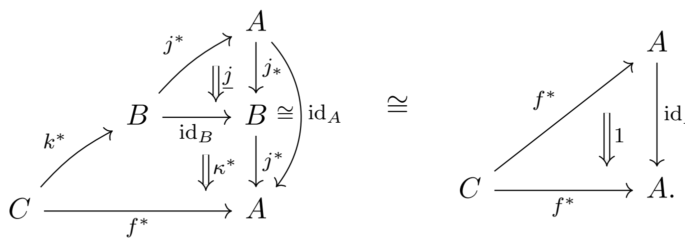
is a right Kan lifting. Since the counit $\underline{j} \colon \mathrm{id}_B \Rightarrow j_* \circ j^*$ of $j_* \dashv j^*$ is an absolute right Kan lifting, so is the 2-cell



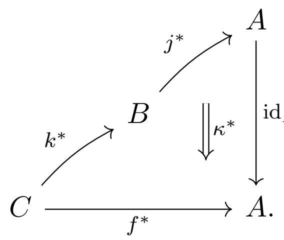
It follows that the composite 2-cell



is a right Kan lifting. Now j is fully faithful so that the unit $\overline{j} \colon \mathrm{id}_A \Rightarrow j^* \circ j_*$ is an isomorphism. Hence the 2-cell above can be seen as a right Kan lifting of f^* along id_A and we have the isomorphism



By the triangle identity equations, the left-hand side becomes



Since this is isomorphic to the identity 2-cell $1 \colon \mathrm{id}_A \circ f^* \Rightarrow f^*$, κ^* is an isomorphism. Therefore κ is also an isomorphism. □

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